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Estimation Techniques for Bilinear Control Systems

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Abstract: The problem of estimating reachable sets of impulsive control systems with uncertainties in initial states and in the matrix of the system is studied. We assume that the initial states to be unknown but belong to a given star-shaped symmetric nondegenerate polytope. The matrix included in the differential equations of the system dynamics is uncertain and only bounds on admissible values of this matrix coefficients are known. We present here the approach that allow to find ellipsoidal estimates of reachable sets which uses the special structure of the bilinear control system. The algorithm of constructing such ellipsoidal set-valued estimates and numerical simulation results are given.

Keywords: Bilinear control systems, Ellipsoidal calculus, Estimation, Set-membership uncertainty.

1. INTRODUCTION

In this paper we deal with the problem of estimation the trajectory tubes of impulsive control systems described by differential equations in the case of set-membership description of uncertain parameters. Control functions in a dynamical system are generalized (impulsive) and are chosen in the class of measures generated by scalar functions of bounded variation at a given time interval. A special kind of nonlinear systems (bilinear systems) where the matrix included in the differential equations of the system dynamics is uncertain was considered. The bounds on admissible perturbations of the matrix are known. Such systems can simulate various mechanical, electrical, and other types of systems whose parameters are unknown, but can vary within certain limits (Boscain et al. (2013); Boussaïd et al. (2013); Nihtila (2010)). As an example, one can indicate mechanical systems in which the stiffness or friction coefficients are given inaccurately. Electrical systems where the resistance, capacitance, inductance, or feedback coefficients are known with a certain accuracy can also be described within the framework of this model.

This system was considered under uncertainty on initial data (Chernousko (1994); Chernousko and Ovseevich (2004); Kurzhanski (1977); Kurzhanski and Valyi (1997); Kurzhanski and Filippova (1993); Pereira and Silva (2000); Polyak B.T. et al. (2004); Schweppe (1973); Walter and Pronzato (1997)). Here it is assumed that the initial state of the system is known not exactly, but is bounded by a given star-shaped set (Mazurenko (2006); Tidmore (1969)). Reachability is well studied for convex sets. However, in concrete applied problems the sets of initial states and admissible states of the system are usually not convex, but have special properties (for example, they are star-shaped). In this case it is useful to have estimates for the exact reachable set. The most developed approaches for estimating reachable set are the method of ellipsoidal calculus (Chernousko (1994); Chernousko and Ovseevich (2004); Kurzhanski and Valyi (1997); Polyak B.T. et al. (2004); Filippova (2012); Filippova and Matviychuk (2015b)) and the method of polyhedral techniques (Kostousova (2012)). This study continuous the researches (Filippova and Matviychuk (2015a); Matviychuk (2016, 2017, 2018)). In this paper, we develop methods of ellipsoidal approximation and present modified estimates of the reachable set of the system using a special structure of the initial set. The external ellipsoidal estimates of reachable sets of such bilinear impulsive control systems are considered here.

2. PROBLEM FORMULATION

In this section, we introduce main necessary notations used in the paper and give the basic formulation of the problem.

Let R^n be the n–dimensional vector space, $\operatorname{comp} R^n$ be the set of all compact subsets of R^n, $\operatorname{conv} R^n$ be the set of all convex and compact subsets of R^n, $R^{n\times n}$ stands for the set of all real $n \times n$–matrices, $\widetilde{R}^{n\times n} \subset R^{n\times n}$ stands for the set of all symmetric positive definite matrices, and $x'y = (x, y) = \sum_{i=1}^{n} x_i y_i$ be the usual inner product of $x, y \in R^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$. Let $I \in R^{n\times n}$ be the identity matrix, $\operatorname{tr}(A)$ be the trace of $n \times n$-matrix A (the sum of its diagonal elements), $\operatorname{diag} b = \operatorname{diag}\{b_i\}$ be the diagonal matrix A with $a_{ii} = b_i$ where b_i are components of the vector b. For a set $A \subset R^n$ we denote its closed convex hull as $\overline{\operatorname{co}}\, A$.

We denote by

$$B(a, r) = \{x \in R^n : \|x - a\| \le r\}$$

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the ball in R^n with center $a \in R^n$ and radius $r > 0$ and by

$$E(a,Q) = \{x \in R^n : \ (Q^{-1}(x-a),(x-a)) \leq 1\}$$

the ellipsoid in R^n with center $a \in R^n$ and symmetric positive definite $n \times n$-matrix Q.

A *parallelepiped* (Kostousova (2012)) $\mathcal{P}(p,P)$ in R^n is a set

$$\mathcal{P}(p,P) = \{x : x = p + \sum_{i=1}^{n} p^i \alpha_i, \ |\alpha_i| \leq 1, \ i = \overline{1,n}\}, \quad (1)$$

where $p \in R^n$ is the center of a parallelepiped, and $P = \{p^1 \ldots p^n\}$ is the orientation matrix ($\det P \neq 0$), p^i are the direction vectors. A parallelepiped with a unit orientation matrix $P = I$ will be called a unit cube.

Consider the following bilinear control system

$$\begin{array}{c} dx = (A(t)x + u(t))dt + B(t)dv, \\ x_0 \in \mathcal{X}_0 \quad t \in [t_0,T], \end{array} \quad (2)$$

where x is the n-dimensional vector of phase coordinates of the system, the matrix $A(\cdot)$ belongs to a set of $n \times n$ real-valued matrices $R^{n\times n}$, vector-function $B(\cdot) \in R^n$ is continuous on $[t_0,T]$.

We assume that $A(t) \in R^{n\times n}$ in (2) has the special form

$$A(t) = A^0(t) + A^1(t), \quad t \in [t_0,T],$$

where the $A^0(t) \in R^{n\times n}$ is given and the measurable matrix $A^1(t) \in R^{n\times n}$ is unknown but bounded,

$$A^1(t) \in \mathcal{A}^1 = \{A = \{a_{ij}\} \in R^{n\times n} : |a_{ij}| \leq c_{ij}, \ i,j = \overline{1,n}\},$$

where $c_{ij} \geq 0$ $(i,j = 1,\ldots n)$ are given, therefore

$$A(t) \in \mathcal{A}(t) = A^0(t) + \mathcal{A}^1, \quad t \in [t_0,T]. \quad (3)$$

Control functions $u(t) \in R^n$ in (2) are assumed Lebesgue measurable on $[t_0,T]$ and satisfying constraint for a.e. $t \in [t_0,T]$

$$u(t) \in \mathcal{U} = E(\hat{a},\hat{Q}). \quad (4)$$

The impulsive control function $v(\cdot) \in R^n$ is a function of bounded variation, monotonically increasing and right-continuous for $t \in [t_0,T]$. Also it is assumed that for some given $\mu > 0$ we have

$$\underset{t\in[t_0,T]}{\mathrm{Var}} \ v(t) = \sup_{\{t_i\}} \sum_{i=1}^{k} |v(t_i) - v(t_{i-1})| \leq \mu, \quad (5)$$

where supremum is taken over all $\{t_i\}$ such that $t_0 \leq t_1 \leq \ldots \leq t_k = T$. Denote by $\mathcal{V}$ the class of all admissible controls $v(\cdot)$ for which (5) holds.

We will assume that the initial value $x_0 = x(-0)$ for the system (2) is unknown but belongs to the set $\mathcal{X}_0 \subset R^n$

$$x_0 \in \mathcal{X}_0 = \mathcal{M}(p), \quad (6)$$

where $\mathcal{M}(p)$ is a symmetric nondegenerate polytope with $2m$ faces. It is assumed that

$$\mathcal{M}(p) = \{x \in R^n : x \in p + \alpha_i \tilde{\mathcal{M}}_i, \ |\alpha_i| \leq 1, \ i = \overline{1,m}\},$$

where $\tilde{\mathcal{M}}_j$ are the faces of the polytope $\mathcal{M}(p)$.

The differential control system (2)–(6) is a model of uncertain dynamic system with a matrix unknown a priori for which only an inclusion description is given $A(t) \in \mathcal{A}$, $x_0 \in \mathcal{X}_0$, $u(t) \in \mathcal{U}$, $v(t) \in \mathcal{V}$.

Let the function $x(\cdot) = x(\cdot; t_0, x_0, A(\cdot), u(\cdot), v(\cdot))$ be a *solution* of the system (2) with initial state $x_0 \in \mathcal{X}_0$, with admissible controls $u(t) \in \mathcal{U}$, $v(t) \in \mathcal{V}$ and with a matrix $A(t) \in \mathcal{A}$. The *trajectory tube* $\mathcal{X}(\cdot)$ of the systems (2)–(6) is defined as the following set

$$\mathcal{X}(\cdot) = \mathcal{X}(\cdot\,; t_0, \mathcal{X}_0, \mathcal{A}, \mathcal{U}, \mathcal{V}) =$$

$$= \bigcup \{x(\cdot) : \ x_0 \in \mathcal{X}_0, \ A(\cdot) \in \mathcal{A}, \ u(\cdot) \in \mathcal{U}, \ v(\cdot) \in \mathcal{V}\}$$

and the *reachable set* is the cross-section $\mathcal{X}(t)$ of this set at the instant t ($t \in [t_0,T]$).

The main problem of the paper is to find external ellipsoidal estimate $E(a^+(t),Q^+(t))$ (with respect to the inclusion of sets) of the reachable set $\mathcal{X}(t)$ ($t_0 < t \leq T$) for studied bilinear systems (2)–(6) by using the analysis of a special type of bilinear control systems with uncertain initial data.

3. MAIN RESULTS

Let us introduse some auxiliary constructions and results.

3.1 Bilinear control system

Consider the following bilinear system

$$\begin{array}{c} \dot{x} = A(t)\,x, \quad x \in R^n, \\ x_0 \in E(a_0,Q_0), \quad A(t) \in \mathcal{A}(t), \quad t_0 \leq t \leq T, \end{array} \quad (7)$$

where the set-valued function $\mathcal{A}(\cdot)$ is defined in (3).

The external ellipsoidal estimate of reachable set $\mathcal{X}(t)$ of the system (7) can be found by applying the following theorem.

Theorem 1. (Chernousko (1996)) Let $a^*(t)$ and $Q^*(t)$ be the solutions of the following system of nonlinear differential equations

$$\dot{a}^* = A^0 a^*, \quad a^*(t_0) = a_0, \quad t \in [t_0,T], \quad (8)$$

$$\dot{Q}^* = A^0 Q^* + Q^* A^{0'} + qQ^* + q^{-1}G, \quad (9)$$

$$Q^*(t_0) = Q_0, \quad t \in [t_0,T],$$

$$q = \big(n^{-1}\,\mathrm{tr}\,((Q^*)^{-1}G)\big)^{1/2},$$

$$G = \mathrm{diag}\Big\{(n-v)\big[\sum_{j=1}^{n} c_{kj}|a_j^*| +$$

$$+\big(\max_{\sigma=\{\sigma_{ij}\}} \sum_{p,q=1}^{n} Q^*_{pq} c_{kp} c_{kq} \sigma_{kp} \sigma_{kq}\big)^{1/2}\big]^2\Big\}, \quad k = \overline{1,n},$$

where the maximum is taken over all $\sigma_{ij} = \pm 1$, such that $c_{ij} > 0$ $(i,j = \overline{1,n})$, and v is a number of such indices k for which we have: $c_{kj} = 0$ for all $j = \overline{1,n}$. Then the following external estimate for the reachable set $\mathcal{X}(t)$ of the system (7) is true

$$\mathcal{X}(t) \subseteq E(a^*(t),Q^*(t)), \quad t_0 < t \leq T.$$

Corollary 2. (Filippova and Matviychuk (2015a)) Under conditions of the Theorem 1 the following inclusion for the reachable set $\mathcal{X}(t)$ of the system (7) holds

$$\mathcal{X}(t_0+\sigma) \subseteq \big(I + \sigma\mathcal{A}(t_0+\sigma)\big)\,E(a_0,Q_0) + o_1(\sigma)B(0,1)$$

$$\subseteq E(a^*(t_0+\sigma),Q^*(t_0+\sigma)) + o_2(\sigma)B(0,1),$$

where $\sigma^{-1}o_i(\sigma) \to 0$ for $\sigma \to +0$ $(i = 1,2)$ and

$$\big(I+\sigma\mathcal{A}(t_0+\sigma)\big)\,\mathcal{X}_0 = \bigcup_{x\in E(a_0,Q_0)} \ \bigcup_{A(\cdot)\in\mathcal{A}(\cdot)} \{x+\sigma A(t_0+\sigma)x\}.$$

3.2 Ellipsoidal estimates of the initial set

Let

$$\tilde{\mathcal{M}}_i \subseteq \mathcal{P}(\tilde{p}_i, \tilde{P}_i), \quad i = \overline{1, m},$$

where $\mathcal{P}(\tilde{p}_i, \tilde{P}_i)$ are the polyhedral estimate of the polytope $\tilde{\mathcal{M}}_i$ (if possible, having a minimum volume among all parallelepipeds that containing $\tilde{\mathcal{M}}_i$). In accordance with (1) the polytope $\tilde{\mathcal{M}}_i$ has form

$$\mathcal{P}(\tilde{p}_i, \tilde{P}_i) = \{x : x = \tilde{p}_i + \sum_{j=1}^{n-1} \tilde{p}_i^j \alpha_j, \ |\alpha_j| \leq 1, \ j = \overline{1, n-1}\},$$

where $\tilde{P}_i = \{\tilde{p}_i^1 \dots \tilde{p}_i^{n-1}\}$, $\tilde{p}_i^j$ are the direction vectors for parallelepipeds $\mathcal{P}(\tilde{p}_i, \tilde{P}_i)$. Note, if the $\mathcal{M}(p)$ is a polytope in R^2, then the $\mathcal{P}(\tilde{p}_i, \tilde{P}_i)$ degenerate into segments (sides of the polygon).

Construct the following parallelepipeds

$$\begin{aligned} \mathcal{P}(p, P_i) = \{x : x = p + \sum_{j=1}^{n-1} \tilde{p}_i^j \alpha_j + (\tilde{p}_i - p)\alpha_n, \\ |\alpha_j| \leq 1, \ j = \overline{1, n}\}, \quad i = \overline{1, m}, \end{aligned} \tag{10}$$

with orientation matrices $P_i = \{p_i^1 \ \dots \ p_i^n\}$, where $p_i^j = \tilde{p}_i^j$, $j = \overline{1, n-1}$, $p_i^n = \tilde{p}_i - p$.

For the construction of ellipsoidal estimates of the initial set we use following lemma.

Lemma 3. For the symmetric nondegenerate polytope $\mathcal{X}_0 = \mathcal{M}(p)$ the following ellipsoidal estimates holds

$$\mathcal{M}(p) \subseteq \bigcup_{i=1}^{m} E(p, D_i^+), \quad D_i^+ = nP_iP_i'. \tag{11}$$

Proof. It is easy to see that for the symmetric nondegenerate polytope $\mathcal{M}(p)$ the following inclusion holds

$$\mathcal{M}(p) \subseteq \bigcup_{i=1}^{m} \mathcal{P}(p, P_i),$$

where $P_i = \{p_i^1 \dots p_i^n\}$, p_i^j are the direction vectors for parallelepipeds $\mathcal{P}(p, P_i)$, defined in (10).

With the help of affine transformation, each parallelepiped $\mathcal{P}(p, P_i)$ can be converted into the unit cube $\mathcal{P}(p, I)$. For the unit cube the estimates $\mathcal{P}(p, I) \subseteq B(p, \sqrt{n})$ are valid. When we return to the original coordinates, we get $\mathcal{P}(p, P_i) \subseteq E(p, nP_iP_i')$. The ellipsoid $E(p, nP_iP_i')$ has a minimum volume among all ellipsoids that containing the $\mathcal{P}(p, P_i)$. Due to (6) we get the estimates (11).

Remark 4. If it is necessary to construct an estimate in the form of a single ellipsoid, then it is sufficient to find an external ellipsoidal estimate for the inclusion (11). Algorithms of external ellipsoidal estimation of the union of the ellipsoids are given in (Vzdornova and Filippova (2006)).

3.3 Bilinear impulsive control system

Consider the bilinear impulsive control system (2) with the following restrictions (3)–(6)

$$dx = \big(A(t)x(t) + u(t)\big)dt + B(t)dv(t),$$

$$x(t_0 - 0) = x_0 \in \mathcal{X}_0 = \mathcal{M}(p), \quad t \in [t_0, T],$$

$$A(t) \in \mathcal{A}, \quad u \in \mathcal{U} = E(\hat{a}, \hat{Q}), \quad v \in \mathcal{V}.$$

We introduce a new time variable (Rishel (1965)) $\eta = \eta(t)$ and a new state coordinate $\tau = \tau(\eta)$

$$\eta(t) = t + \int_{t_0}^{t} dv(s), \quad \tau(\eta) = \inf\{t \ : \ \eta(t) \geq \eta\}$$

and consider the following differential inclusion

$$\frac{d}{d\eta} \begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z), \tag{12}$$

$$z(t_0) = z_0 \in \mathcal{X}_0 = \mathcal{M}(p),$$

$$\tau(t_0) = t_0, \quad t_0 \leq \eta \leq T + \mu,$$

$$H(\tau, z) = \bigcup_{0 \leq \nu \leq 1} \left\{ (1 - \nu) \begin{pmatrix} A(\tau)z + E(\hat{a}, \hat{Q}) \\ 1 \end{pmatrix} + \nu \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix} \right\}.$$

Denote by $w = \{z, \tau\}$ the extended state vector of the differential inclusion (12) and by $W(\eta) = W(\eta; t_0, \mathcal{X}_0 \times \{t_0\}, \mathcal{A})$ $(t_0 \leq \eta \leq T + \mu)$ the reachable set of the system (12).

Theorem 5. For any $\sigma > 0$ the following inclusion holds

$$W(t_0 + \sigma) \subseteq \bigcup_{i=1}^{m} W_i(t_0, \sigma) + o(\sigma)B^{n+1}(0, 1),$$

$$\lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0,$$

$$W_i(t_0, \sigma) = \bigcup_{0 \leq \nu \leq 1} W_i(t_0, \sigma, \nu),$$

$$W_i(t_0, \sigma, \nu) = \begin{pmatrix} E\big(a_i^+(t_0, \sigma, \nu), Q_i^+(t_0, \sigma, \nu)\big) \\ t_0 + \sigma(1 - \nu) \end{pmatrix},$$

$$a_i^+(t_0, \sigma, \nu) = \tilde{a}_i(\sigma, \nu) + \sigma(1 - \nu)\hat{a} + \sigma\nu B(t_0),$$

$$Q_i^+(t_0, \sigma, \nu) = (q^{-1} + 1)\tilde{Q}_i(\sigma, \nu) + (q + 1)\sigma^2(1 - \nu)^2\hat{Q},$$

where functions $\tilde{a}_i(\sigma, \nu)$ and $\tilde{Q}_i(\sigma, \nu)$ are calculated as $a^*(t)$ and $Q^*(t)$ in the Theorem 1 but when we replace matrix A^0 and Q_0 in (8), (9) by $\tilde{A}^0 = (1 - \nu)A^0$ and $Q_0 = nP_iP_i'$, respectively, and $a^*(t_0) = p$. The matrices P_i is defined by the Lemma 3. Here $q = q(\sigma, \nu)$ is the unique positive root of the equation $\sum_{k=1}^{n} 1/q + \lambda_k = n/q(q + 1)$, where $\lambda_k = \lambda_k(\sigma, \nu) \geq 0$ are the roots of the equation $|\tilde{Q}_i(\sigma, \nu) - \lambda\sigma^2(1-\nu)^2\hat{Q}| = 0$.

Proof. The proof of this theorem uses the procedure of external ellipsoidal estimating a sum of two ellipsoids (Chernousko (1994); Kurzhanski and Valyi (1997)). Applying the scheme from (Filippova and Matviychuk (2011, 2015a)). The results of the Theorem 1 and Lemma 3 allow us to find the upper estimates for reachable sets $W(t_0 + \sigma)$ of the differential inclusion (12) in a form of the union of m ellipsoids.

Remark 6. To find the estimate of the reachable set $W(t_0+\sigma)$ we introduce a small parameter $\varepsilon > 0$ and embed the degenerate ellipsoid $W(t_0,\sigma,\nu)$ in the nondegenerate ellipsoid $E\big(w_\varepsilon(t_0,\sigma,\nu),O_\varepsilon(t_0,\sigma,\nu)\big)$:

$$W_i(t_0,\sigma,\nu)\subseteq E\big(w_{\varepsilon,i}(t_0,\sigma,\nu),O_{\varepsilon,i}(t_0,\sigma,\nu)\big),$$

$$w_{\varepsilon,i}(t_0,\sigma,\nu)=\begin{pmatrix} a_i^+(t_0,\sigma,\nu)\\ t_0+\sigma(1-\nu)\end{pmatrix},$$

$$O_{\varepsilon,i}(t_0,\sigma,\nu)=\begin{pmatrix} Q_i^+(t_0,\sigma,\nu) & 0\\ 0 & \varepsilon^2\end{pmatrix}.$$

Thus, for all small $\varepsilon>0$ we get

$$W(t_0+\sigma)\subseteq\bigcup_{i=1}^m W_i(t_0,\sigma)\subseteq\bigcup_{i=1}^m W_{\varepsilon,i}(t_0,\sigma),$$

$$W_{\varepsilon,i}(t_0,\sigma)=\bigcup_{0\le\nu\le1} E\big(w_{\varepsilon,i}(t_0,\sigma,\nu),O_{\varepsilon,i}(t_0,\sigma,\nu)\big),$$

where $\lim\limits_{\varepsilon\to+0} h(W_i(t_0,\sigma),W_{\varepsilon,i}(t_0,\sigma))=0.$

The passage to the family of nondegenerate ellipsoids enables one to use the algorithms of (Vzdornova and Filippova (2006); Filippova and Matviychuk (2011)) and construct an external estimate of the union of the ellipsoids

$$W_{\varepsilon,i}(t_0,\sigma)\subset E_{\varepsilon,i}(w^+(\sigma),O^+(\sigma)).$$

The following lemma explains the reason of construction of the auxiliary differential inclusion (12).

Lemma 7. (Filippova and Matviychuk (2011)) The reachable set $\mathcal{X}(T)$ is the projection of $W(T+\mu)$ at the subspace of variables z: $\mathcal{X}(T)=\pi_z W(T+\mu)$.

The next iterative algorithm is based on Theorem 5 and allows to find the external ellipsoidal estimates of the reachable sets of the studied bilinear impulsive control system (2)–(6).

Algorithm. Subdivide the time segment $[t_0,T+\mu]$ into subsegments $[t_j,t_{j+1}]$ where $t_j=t_0+j\sigma$ $(j=\overline{1,s})$, $\sigma=(T+\mu-t_0)/s$, $t_s=T+\mu$. Also subdivide the segment $[0,1]$ into subsegments $[\nu_k,\nu_{k+1}]$ where $\nu_k=kh_*$, $h_*=1/k$, $\nu_0=0$, $\nu_l=1$ $(k=\overline{0,l})$.

(1) For the given symmetric nondegenerate polytope $\mathcal{X}_0=\mathcal{M}(p)$ find m ellipsoids $E(p,D_i^+)$, $i=\overline{1,m}$ by Lemma 3.
(2) For each ellipsoid $E(p,D_i^+)$ $(i=\overline{1,m})$ define the sets $W_i(t_0,\sigma,\nu_k)$, $k=\overline{0,l}$ by Theorem 5.
(3) Fix the small parameter $\varepsilon>0$ and for sets $W_i(\sigma,\nu_k)$ $(k=\overline{0,l})$ find ellipsoids $E(w_{\varepsilon,i}(t_0,\sigma,\nu_k),O_{\varepsilon,i}(t_0,\sigma,\nu_k))$ by Remark 6.
(4) Find ellipsoid $E_\varepsilon(w_1(\sigma),O_1(\sigma))$ in R^{n+1} such that

$$W_{\varepsilon,i}(t_0,\sigma)=\bigcup_{k=1}^{l} E\big(w_{\varepsilon,i}(t_0,\sigma,\nu_k),O_{\varepsilon,i}(t_0,\sigma,\nu_k)\big)\subseteq$$
$$\subseteq E_{\varepsilon,i}(w_1(\sigma),O_1(\sigma)),\quad i=\overline{1,m}.$$

At this step we find the ellipsoidal estimate for the union of a finite family of ellipsoids (Filippova and Matviychuk (2011)).
(5) Find the projection of $E(w^1_{\varepsilon,i}(\sigma),O^1_{\varepsilon,i}(\sigma))$ at the subspace of variables z by Lemma 1: $E(a_i^1,Q_i^1)=\pi_z E(w^1_{\varepsilon,i}(\sigma),O^1_{\varepsilon,i}(\sigma))$.
(6) Consider the system on the next subsegment $[t_1,t_2]$ with $E(w^1_{\varepsilon,i}(\sigma),O^1_{\varepsilon,i}(\sigma))$ as the initial ellipsoids at instant t_1.
(7) The next step repeats the previous iterations. If $t=T+\mu$ then end of the procedure.

At the end of the process we will get the external estimate of the reachable set $\mathcal{X}(T)$ of the impulsive control system (2)–(6) with uncertain matrix of the system and uncertain initial state basing on the special structure of the data $\mathcal{A},\mathcal{U}$ and $\mathcal{X}_0$ in a form of the union of m ellipsoids

$$\mathcal{X}(T)\subseteq\bigcup_{i=1}^m E(a_i^+(T),Q_i^+(T)).$$

3.4 Numerical Simulation

The following example illustrates the main result of the study.

Example 1. Consider the following bilinear control system $(t_0\le t\le T)$

$$\begin{cases}\dot x_1 &= x_2+u_1,\\ \dot x_2 &= c(t)x_1+u_2.\end{cases}$$

Here $t_0=0$, $T=0.85$, $x_0\in\mathcal{X}_0$ and $\mathcal{U}=B(0,1)$, the uncertain but bounded measurable function $c(t)$ satisfies the inequality $|c(t)|\le1$ $(t_0\le t\le T)$. Here $\mathcal{X}_0$ is the symmetric polytope $\mathcal{M}(0)$ with vertices: $(1.5,0)$, $(0.5,0.5)$, $(0,1.5)$, $(-0.5,0.5)$, $(-1.5,0)$, $(-0.5,-0.5)$, $(0,-1.5)$, $(0.5,-0.5)$.

For $\mathcal{M}(0)$ parallelepipeds $\mathcal{P}(0,P_i)$, $i=1,2,3,4$ were constructed (Fig. 1),

$$\mathcal{X}_0=\mathcal{M}(0)=\bigcup_{i=1}^4\mathcal{P}(0,P_i),$$

$$P_1=\begin{pmatrix}-0.25 & 0.25\\ 1 & 0.5\end{pmatrix},\quad P_2=\begin{pmatrix}0.25 & 0.25\\ 1 & -0.5\end{pmatrix},$$

$$P_3=\begin{pmatrix}1 & -0.5\\ 0.25 & 0.25\end{pmatrix},\quad P_4=\begin{pmatrix}-1 & 0.5\\ 0.25 & 0.25\end{pmatrix}.$$

The external ellipsoidal estimates for $\mathcal{X}_0$ are given in Fig. 2. The trajectory tube $\mathcal{X}(t)$ is shown in the Fig. 3. This tube was constructed approximately with using results (Ushakov et al. (2012); Matviychuk and Matviychuk (2018)) The external ellipsoidal estimates $E(a_i^+(t),Q_i^+(t))$ are given in Fig. 4.

4. CONCLUSION

The paper deals with the problem of state estimation for uncertain impulsive control systems. Here we assume that the initial state is unknown but belongs to a given star-shaped symmetric nondegenerate polytope and the matrix in the linear part of the system is also unknown but bounded. Basing on results of ellipsoidal calculus developed earlier for some classes of uncertain systems we present the modified state estimation approach which uses the uncertainty of special structure in the control system. This approach allows to construct the external ellipsoidal estimates of reachable sets.

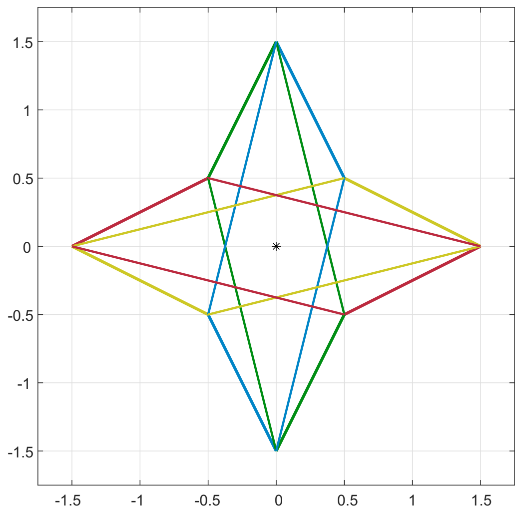

Fig. 1. The initial set $\mathcal{X}_0 = \mathcal{M}(0)$.

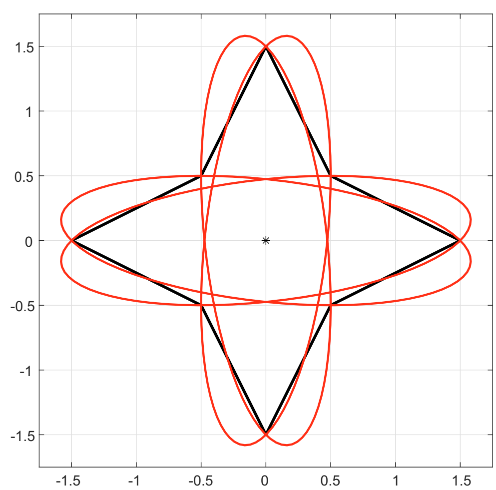

Fig. 2. The external estimates of the initial set $\mathcal{X}_0$.

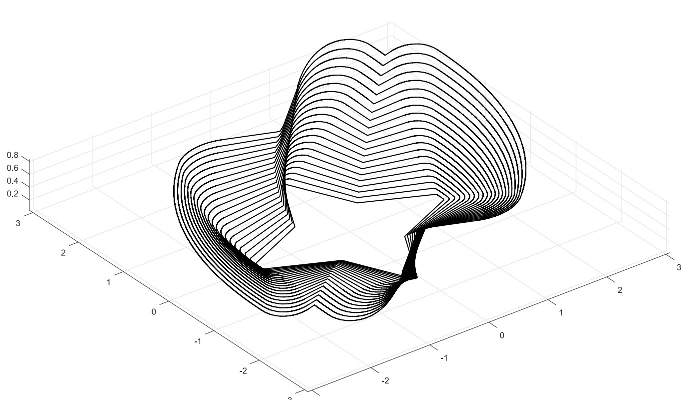

Fig. 3. The trajectory tube $\mathcal{X}(t)$, $0 < t \leq 0.85$.

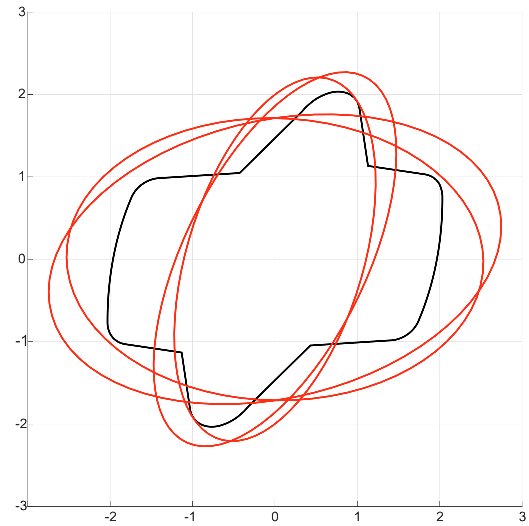

Fig. 4. Ellipsoidal estimates of the reachable set $\mathcal{X}(0.4)$.

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